Advanced Topics in Random Graphs
Exercise Sheet 2

Question 1. Let G be a graph with ϵn^2 edges for some $\epsilon > 0$. Using the dependent random choice lemma from the lectures, show that G contains a 1-division of K_p for some $p = \epsilon^{\frac{3}{2}} n^{\frac{1}{2}}$.

(*Improve the above to $p = \epsilon n^{\frac{1}{2}}$ and show that this is sharp.)

Question 2. Let G be the graph with $V(G) = \{0,1\}^m$ where x and y are adjacent if they differ in more than $\frac{m}{2}$ coordinates and let $\epsilon > 0$ be a positive constant.

Show that there is no set $U \subset V$ with $|U| \geq \epsilon |V|$ such that every pair of vertices in U has $\geq \epsilon n$ many common neighbours.

(You may assume that every set of size $|U| \geq \sum_{i=1}^{t} \binom{m}{i}$ contains two points differing in at least $2t+1$ coordinates)

Question 3. Let $\epsilon > 0$ be a positive constant and $r \leq n$ be integers. Suppose G is a graph on $N > 4r\epsilon^{-r} n$ vertices which has at least $\epsilon \frac{N}{2}$ edges. Show that there is a subset $U \subseteq V(G)$ of size $|U| > 2n$ such that the number of subsets $S \subset U$ with $|S| = r$ and less than n common neighbours is at most

$$\frac{1}{(2r)^r}\binom{|U|}{r}.$$

Question 4. Let H be a bipartite graph with maximum degree r. Show that if G contains a subset U satisfying the conclusion of Question 3, then G contains a copy of H.

(Hint : Call sets $S \subseteq U$ *good* if they are contained in the right number of sets of size d with at least n common neighbours. Try to embed the graph so that the neighbourhoods of the vertices in the right partition class are always good)

Question 5. Suppose that H is bipartite on n vertices and has maximum degree r. Show that the Ramsey number of H is at most $r2^{r+3}n$.